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On stability of exterior stationary Navier-Stokes flows

Tetsuro MIYAKAWA (宮川 鉄朗)

Department of Applied Science, Faculty of Engineering
Kyushu University

We consider the existence and stability of solutions of the steady incompressible Navier-Stokes equations in an exterior domain Ω of $\mathbb{R}^n$ $(n \geq 3)$:

$$\text{(S)} \qquad \begin{aligned} -\Delta w + w \cdot \nabla w = f - \nabla q \quad & \text{in} \quad \Omega \\ \nabla \cdot w = 0 \qquad\qquad & \text{in} \quad \Omega \\ w|_\Gamma = w^*, \quad \lim_{|x|\to\infty} w = 0 & \end{aligned}$$

for prescribed external force $f = (f_1, \cdots, f_n)$ and boundary data w^*. Here, w and p_0 denote, respectively, unknown velocity and pressure ; and Γ is the (smooth) boundary of Ω. Throughout this note we assume, without loss of generality, that $0 \notin \overline{\Omega}$.

Problem (S) was systematically studied for the first time by J. Leray [10] in case $n = 3$ within the framework of the so-called weak solutions having finite Dirichlet integrals. He proved existence of at least one weak solution for general data. His solutions are smooth, but their uniqueness remains open even in the case of small data. Finn [5] discussed the existence of small solutions and proved among others that if w^* is smooth, if $f = 0$, and if w^* is small in an appropriate sense, then problem (S) possesses a smooth solution w with the property that

$$\text{(1)} \qquad |w| \leq C|x|^{-1}, \qquad |\nabla w| \leq C|x|^{-2} \log |x|.$$

Our first result stated below improves (1) and generalizes it to the case of higher space dimensions $n \geq 4$. Namely, one can show the following

Theorem 1. *Suppose that $f = (f_1, \cdots, f_n)$ is written in the form*

$$f_j = \sum_{k=1}^{n} \partial_k F_{jk} \qquad (j = 1, \ldots, n),$$

by means of some smooth functions $F = (F_{jk})$, satisfying

$$|F| \le C_1|x|^{1-\mu}, \qquad |\nabla F| \le C_1|x|^{-\mu}$$

for some constant $\mu \ge 3$. If $w^* \in C^{2+\alpha}(\Gamma)$ for some $0 < \alpha < 1$, and if $\|w^*\|_{C^{2+\alpha}}$ and C_1 are sufficiently small, then problem (S) *possesses a solution w such that*

$$(2) \qquad |w| \le \begin{cases} C|x|^{2-\mu} & (3 \le \mu \le n) \\ C|x|^{2-n} & (\mu > n) \end{cases} \qquad |\nabla w| \le \begin{cases} C|x|^{1-\mu} & (3 \le \mu \le n) \\ C|x|^{1-n} & (\mu > n) \end{cases}$$

To show Theorem 1, we transform problem (P) into the integral equation

$$\text{(IE)} \qquad \begin{aligned} w &= \Phi(w) \\ &\equiv E \cdot (f - w \cdot \nabla w) + \int_\Gamma (\varphi \cdot T[E,Q]\nu + E \cdot h)\, dS, \end{aligned}$$

where

$$E \cdot (f - w \cdot \nabla w) = \int_\Omega E(x-y)(f - w \cdot \nabla w)(y)dy.$$

Here, $E = (E_{jk})$ and $Q = (Q_j)$ are the Stokes fundamental solution tensor and $T[E,Q] \cdot \nu$ is the associated normal stress (see [9]). Given f and w^*, equation (IE) for unknown functions φ and h can be solved uniquely, as shown in [9]. The right-hand side of (IE) is then estimated in appropriate function spaces by applying the Schauder estimates of [16] and the method of [12] for estimating the volume potentials. Theorem 1 is then deduced via the contraction mapping principle. It should be emphasized here that if $n = 3$ or if $n \ge 4$, $\mu = n$, the estimate given in [12] is indispensable in order to deduce the desired estimate for volume potentials. Indeed, in the other cases one can apply the well-known result

$$\int |x-y|^{-\alpha}(1+|y|)^{-\beta}dy \le \begin{cases} C|x|^{n-\alpha-\beta} & (0 < \beta < n), \\ C|x|^{-\alpha} & (\beta > n), \end{cases}$$

which holds for $0 < \alpha < n$ and $\alpha + \beta > n$, in order to estimate the volume potentials.

Theorem 1 shows in particular that the factor $\log|x|$ is redundant in (1), and this fact enables us to discuss the stability of solutions w obtained there. To this end, suppose we are given a disturbance of w of the form $w + a$, and let $u = u(t)$ denote the time-evolution of a, which is governed by the initial value problem

$$\text{(P)} \qquad \begin{aligned} &\frac{\partial u}{\partial t} + w \cdot \nabla u + u \cdot \nabla w = \Delta u - \nabla p \quad (x \in \Omega, \ t > 0) \\ &\nabla \cdot u = 0 \qquad\qquad\qquad\qquad\qquad (x \in \Omega, \ t \ge 0) \\ &u|t = 0 = a, \quad u|_\Gamma = 0, \\ &\lim_{|x| \to \infty} u = 0. \end{aligned}$$

Our first stability result concerns large time behavior in L^2 of global (in time) weak solutions to problem (P), which always exist for arbitrary initial data a in L^2.

Theorem 2. (i) *Let w be a solution of* (S) *given in Theorem 1. Then there exists a constant $0 < C_n \leq (n-2)/2$ such that if*

$$\sup |x| \cdot |w(x)| < C_n$$

then problem (P) *has a weak solution u which goes to 0 in L^2 as $t \to \infty$.*

(ii) *If a solution $u^0(t)$ to the linearized equation of* (P) *with $u^0(0) = a$ decays like $t^{-\alpha}$ for some $\alpha > 0$, then the weak solution u treated in* (i) *decays in the following way :*

$$\|u(t)\|_2 = O(t^{-\beta}), \quad \beta = \min\{\alpha, n/4 - \varepsilon\}, \quad \forall \varepsilon > 0, \tag{3}$$

where $\|\cdot\|_2$ is the L^2-norm.

(iii) *If $n \geq 4$ and if w satisfies the additional property :*

$$\nabla w \in L^r \quad \text{for some} \quad n/(n-1) < r < n/2, \tag{4}$$

then one can take $\varepsilon = 0$ in (3)*. The same is true in the case $n = 3$ if*

$$\nabla w \in L^r \quad \text{for some} \quad 1 < r < 3/2. \tag{4$'$}$$

Theorem 2 extends the decay result of [1] which was obtained for weak solutions of problem (P) with $w = 0$. The method of proof is basically the same as those given in [1,13,14]. Observe that if $n \geq 4$, then $n/(n-1) < n/2$; so Theorem 1 ensures existence of a solution w of (S) satisfying (4). On the other hand, if $n = 3$, then $n/(n-1) = n/2 = 3/2$; so (1) and (4)$'$ together imply

$$\nabla w \in L^{n/(n-1)} \quad \text{so that} \quad w \in L^{n/(n-2)}. \tag{5}$$

The next result shows, however, that the stationary solutions w satisfying (5) exist in a very restrictive situation :

Theorem 3. (i) *Under the condition* (1.4), *we have*

$$\int_\Gamma \nu \cdot (T[w,q] - w^* \otimes w^* + F) = 0 \tag{6}$$

together with the associated pressure q. Here ν is the unit outward normal to Γ and

$$T[w,q] = (T_{jk}[w,q])_{j,k=1}^n, \qquad T_{jk}[w,q] = -\delta_{jk} q + \partial_j w_k + \partial_k w_j.$$

(ii) *If $n \geq 4$, and if w satisfies* (6), *then $\nabla w \in L^r$ for all $1 < r \leq \infty$. The same is true for the case $n = 3$ if in addition $\nabla w \in L^s$ for some $1 < s < 3/2$.*

Our final result discusses large time behavior of strong solutions of (P), which exist for small initial data a in L^n.

Theorem 4. *For each $1 < r < n$ there is a number $\mu = \mu(n, r) > 0$ such that if a solution w of* (S) *satisfies*

$$\sup |x| \cdot |w(x)| + \sup |x|^2 \cdot |\nabla w(x)| \leq \mu,$$

then we have the following :

(i) *For each $a \in L^n \cap L^r$, which is small in L^n, there exists a unique smooth solution u defined for all $t \geq 0$ such that*

$$\|u(t)\|_\infty = O(t^{\varepsilon - n/2r}), \quad \forall \varepsilon > 0, \tag{7}$$

where $\|\cdot\|_\infty$ is the L^∞-norm.

(ii) *If w satisfies* (4) *and* (4)′, *respectively, then one can take $\varepsilon = 0$ in* (7). *In particular, the strong solutions of the Navier-Stokes equations (i.e., problem* (P) *with $w = 0$) have the decay property*

$$\|u(t)\|_\infty = O(t^{-n/2r}).$$

Theorem 4 improves the decay results given in [6,8,11]. One can also take the initial data a not from the usual Lebesgue space L^r but from the *weak* space L^r_w as described for instance in [15].

In showing Theorem 2 (i), (ii), and Theorem 4 (i), a crucial role is played by the decay properties of the semigroup (in general L^r spaces) generated by the linearized operator

$$L = A + B, \qquad Bu = P(w \cdot \nabla u + u \cdot \nabla w),$$

where P is the bounded projection onto the space of solenoidal L^r vector fields and $A = -P\Delta$ is the Stokes operator. The decay properties of the semigroup generated by A are discussed in detail in [1,3,4,7]. Due to the decay property (2) of the stationary solutions w, one can apply the standard Neumann series expansion to the resolvent of L, to deduce various time-decay properties of the corresponding semigroup. To show Theorem 2 (iii) and Theorem 4 (ii), we apply a perturbation argument to the result of [3] which asserts that the Stokes semigroup maps L^r vector fields ($1 < r < \infty$) to L^∞ fields ; to do so, one needs to assume the more stringent conditions for ∇w as stated in (4) and (4)′.

The complete proofs of Theorems 1–4 are given in [2] ; so the details are omitted here. Recently, Kozono [17] has announced that Theorem 4 (i) can be deduced with no smallness assumption on the size of $\sup |x|^2 |\nabla w(x)|$.

We finally note that the condition

$$|w| \leq C|x|^{-1}, \qquad |\nabla w| \leq C|x|^{-2}, \tag{8}$$

which is always satisfied by our stationary solutions, is considered as stable under the time-evolution, provided we interpret (8) as

$$|w| \in L^n_w, \qquad |\nabla w| \in L^{n/2}_w, \tag{8$'$}$$

with the fact that $|x|^{-1} \in L^n_w$ and $|x|^{-2} \in L^{n/2}_w$ in mind. Indeed, one can show the following

Theorem 5. *Suppose that $a \in L^n_w$, $\nabla a \in L^{n/2}_w$, and $a|_\Gamma = 0$. If a is small in L^n_w, then there exists a unique strong solution u of* (P) *such that*

$$|u(t)| \in L^n_w, \qquad |\nabla u(t)| \in L^{n/2}_w.$$

Theorem 5 is completely proved in [2] by systematically applying the real interpolation method to the semigroup generated by the Stokes operator A ; so the details are omitted here.

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